

Comments to the S-matrix interpretation of finite-temperature field theories

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Abstract

It is shown that there is the possibility to find at least in the perturbation framework the Matsubara theory from the S -matrix interpretation of the real-time finite-temperature field theory if the system under consideration is in an equilibrium state.

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1 Introduction

In this paper we will discuss the S -matrix interpretation of the real-time finite temperature field theory [1] in connection with the Matsubara imaginary time theory [2]. We shall show the qualitative conditions in frame of which both theories are coincide.

It will be seen that the main condition is the state of equilibrium. We will define the equilibrium state as a state with Gaussian fluctuations of the thermodynamical parameters. This is evident for Matsubara theory. But in the S -matrix formalism it gives the nontrivial constraints since such quantity as, for instance, the temperature introduced in it as the Lagrange multiplier, so as in the microcanonical formalism.

We will assume also that the perturbation theory is applicable. The importance of this condition will be discussed in Sec.3.

It will be shown in result that under this conditions the S -matrix theory can be analytically continued to the Matsubara imaginary time theory (Sec.2). This result seems important since it shows up the ability of the S -matrix approach to describe the statistical systems.

The introduction of the temperature T as the Lagrange multiplier means that the parameter T introduced in a theory for sake of “economy” description of the many-particle system. In Sec.3 we find the quantitative conditions when such simplification is valid. We will show that T is a “good” parameter if there is asymptotic connections between various energy correlation functions.

2 The S -matrix finite-temperature field theory

The experience of papers [1, 3] shows that the Wigner functions [4] generating functional [5] R_S can be written in the S -matrix framework in a factorised form independently from boundary conditions:

$$R_S(\beta_+, \beta_-) = e^{N(\hat{\phi}_i^* \hat{\phi}_j; \beta_+, \beta_-)} R_0(\phi_{\pm}), \quad (2.1)$$

where N is some nonlinear operator of

$$\hat{\phi}_i(q) = \int dx e^{-iqx} \delta / \delta \phi_i(x), i = +, -. \quad (2.2)$$

The concrete form of this operator depends from boundary conditions, i.e. from the environment of the system. For the case of uniform temperature distribution,

$$N(\hat{\phi}_i^* \hat{\phi}_j; \beta_+, \beta_-) = \int d\omega(q) \hat{\phi}_i^*(q) n_{ij}(q, (\beta)) \hat{\phi}_j(q) + \dots, \\ d\omega(q) = \frac{d^3 q}{(2\pi)^3 2\epsilon(q)}, \epsilon(q) = (q^2 + m^2)^{1/2}. \quad (2.3)$$

(The summation over all configurations of (ij) is assumed.) Here $n_{ij}(q)$ is the occupation number. Assuming that the environment is composed from noncorrelated particles one

can calculate [1] that

$$\begin{aligned} n_{++} &= n_{--} = \tilde{n}(|q_0|(\beta_+ + \beta_-)/2), \\ n_{+-} &= \Theta(q_0)(1 + \tilde{n}(q_0\beta_+)) + \Theta(-q_0)\tilde{n}(-q_0\beta_-), \\ n_{-+} &= \Theta(q_0)\tilde{n}(q_0\beta_+) + \Theta(-q_0)(1 + \tilde{n}(-q_0\beta_-)), \end{aligned} \quad (2.4)$$

where

$$\tilde{n}(q_0\beta) = \frac{1}{e^{\beta q_0} - 1}, q_0 = (q^2 + m^2)^{1/2} \quad (2.5)$$

is the mean multiplicity of environment particles with energy q_0 at a given temperature $1/\beta$. The subsequent terms in (2.3) describes the correlations among particles of the environment. In this formulation of the theory they are the free parameters. We can consider only the first term assuming absence of correlations.

We shell interpret $1/\beta_-$ as the temperature of initial state and $1/\beta_+$ as the final state temperature [1]. Therefore, by definition, the S -matrix theory is the two-temperature.

The generating functional

$$R_0(\phi_\pm) = Z(\phi_+)Z(\phi_-), \quad (2.6)$$

where $Z(\phi_\pm)$ are the vacuum into vacuum transition amplitudes:

$$Z(\phi_\pm) = \int D\Phi_\pm e^{\pm iS_0(\Phi_\pm) \mp iV(\Phi_\pm + \phi_\pm)}. \quad (2.7)$$

They were defined in [1] on the Mills' time contour [6]:

$$\Phi_\pm = \Phi_\pm(t \in C_\pm), C_\pm : t \rightarrow t \pm i\varepsilon, \varepsilon \rightarrow +0, -\infty \leq t \leq +\infty. \quad (2.8)$$

In eq.(2.7) S_0 is the free part of the action and V describes the interactions. Calculating integral (2.7) the usual field-theoretical boundary conditions:

$$\int_{\sigma_\infty} d\sigma_\mu \Phi_\pm \partial^\mu \Phi_\pm = 0 \quad (2.9)$$

must be applied since the environment is fixed by the operator N . It means that (σ_∞ is the infinitely far hypersurface)

$$\Phi_\pm(t = -\infty) = 0, \Phi_\pm(t = +\infty) = 0 \quad (2.10)$$

(the space boundary conditions are trivial). In the perturbation theory framework

$$Z(\phi_+) = e^{-i \int dx \hat{j}_+(x) \hat{\Phi}_+(x)} e^{-iV(\Phi_+ + \phi_+)} e^{-\frac{i}{2} \int dx dx' j_+(x) D_{++}(x-x') j_+(x')}, \quad (2.11)$$

where $D_{++}(x - x')$ is the causal (Feynman) Green function and $\hat{j}(x) \equiv \delta/\delta j(x)$, $\hat{\Phi}(x) \equiv \delta/\delta \Phi(x)$.

So, we see that $R_0(\phi_\pm)$ describes pure fields dynamics and $N(\hat{\phi}_i^* \hat{\phi}_j; \beta_+, \beta_-)$ contains the thermodynamics. One can say that the action of operator $\exp\{N\}$ maps the interacting fields asystem on the thermodynamical state.

Using (2.1), (2.3) and (2.11) we can find that

$$R_S(\beta_+, \beta_-) = e^{-iV(-i\hat{j}_+) + iV(-i\hat{j}_-)} e^{\frac{i}{2} \int dx dx' j_i(x) G_{ij}(x-x', (\beta)) j_j(x')}, \quad (2.12)$$

where the n_{ij} dependence was absorbed in the Green functions G_{ij} . In the momentum representation the Green functions looks as follows:

$$i\tilde{G}(q, (\beta)) = \begin{pmatrix} \frac{i}{q^2 - m^2 - i\varepsilon} & 0 \\ 0 & -\frac{i}{q^2 - m^2 - i\varepsilon} \end{pmatrix} + 2\pi\delta(q^2 - m^2) \begin{pmatrix} \tilde{n}(\frac{\beta_+ + \beta_-}{2} | q_0|) & \tilde{n}(\beta_+ | q_0|) a_+(\beta_+) \\ \tilde{n}(\beta_- | q_0|) a_-(\beta_-) & \tilde{n}(\frac{\beta_+ + \beta_-}{2} | q_0|) \end{pmatrix}, \quad (2.13)$$

and

$$a_{\pm}(\beta) = -e^{\beta(|q_0| \pm q_0)/2} \quad (2.14)$$

The matrix Green functions was introduced firstly in [7].

Let us assume now that the energy E of the system is fixed and, therefore, β_{\pm} are the fluctuating quantities. The corresponding generating functional $r(E)$ can be deduced calculating the integrals [1]:

$$r(E) = \int \frac{d\beta_+}{2\pi i} \frac{d\beta_-}{2\pi i} e^{E(\beta_+ + \beta_-) - F(\beta_+, \beta_-)}, \quad (2.15)$$

where

$$F(\beta_+, \beta_-) \equiv -\ln R_S(\beta_+, \beta_-) \quad (2.16)$$

and the integrations are performed along the imaginary axis. This integrals we shall compute by the stationary phase method. In our case there are two equations of state (we distinguish the initial and final states temperatures):

$$E = \frac{\partial}{\partial \beta_i} F(\beta_+, \beta_-), i = +, -. \quad (2.17)$$

In absence of the energy dissipation this two equations have the same solution:

$$\beta_{\pm} = \beta(E) > 0, Im\beta = 0. \quad (2.18)$$

Expanding integrals in (2.15) over $(\beta_+ - \beta)$, $(\beta_- - \beta)$ we shall leave onle first term. This imply that the fluctuations of β_+ and β_- near $\beta(E)$ are Gaussian. This question will be discussed also in Sec.3.

In result, the generating functional in the energy representation is

$$r(E) \sim R_S(\beta_+ = \beta, \beta_- = \beta) |det \left(\frac{\partial^2}{\partial \beta_i \partial \beta_j} F(\beta_+, \beta_-) \right)|_{\beta_{\pm} = \beta}^{-1/2} e^{2\beta(E)E}. \quad (2.19)$$

It was assumed that

$$det \left(\frac{\partial^2}{\partial \beta_i \partial \beta_j} F(\beta_+, \beta_-) \right) \neq 0. \quad (2.20)$$

So, the generating functional in the temperature representation for equilibrium case is

$$R_S(\beta) = R_S(\beta_+ = \beta, \beta_- = \beta). \quad (2.21)$$

We can compare now $R_S(\beta)$, defined in (2.12), with the Niemi-Semenoff's generating functional [8] R_{NS} .

The path integral representation for generating functional R_{NS} has the form [8]:

$$R_{NS}(\beta) = \int D\Phi e^{iS_{C_\beta(T_-, T_+)}(\Phi, j)} \quad (2.22)$$

where the total action includes an external source j :

$$S_{C_\beta(T_-, T_+)}(\Phi, j) = S_{C_\beta(T_-, T_+)}(\Phi) + \int_{C_\beta(T_-, T_+)} dx j(x) \Phi(x). \quad (2.23)$$

It is remarkable that the integration performed in (2.22) over fields defined on the Mills' closed-time contour. So,

$$C_\beta(T_-, T_+) = C_1(T_-, T_+) + C_2(T_+, T_+ - i(\beta + \alpha)/2) + \\ + C_3(T_+ - i(\beta + \alpha)/2, T_- - i(\beta + \alpha)/2) + C_4(T_- - i(\beta + \alpha)/2, T_- - i\beta). \quad (2.24)$$

This means that the time integral in (2.23) starts from the point $T_- + i\varepsilon$, $\varepsilon \rightarrow +0$, and goes to $T_+ + i\varepsilon$, $T_+ > T_-$. It is the C_1 part of C_β . Contour C_2 starts from $T_+ + i\varepsilon$ and goes to $T_+ - i(\beta + \alpha)/2$, etc. The times T_- , T_+ can be chosen arbitrarily. So, if $T_+ = T_-$ the contour $C_\beta(T_-, T_-)$ will coincide with Matsubara time contour [2].

Note also that the theory has the remarkable degree of freedom [9] which follows from the translational invariance of the system. One can choose arbitrary α in the interval:

$$-\beta \leq \alpha \leq \beta. \quad (2.25)$$

This gives the possibility to assume that there are no singularities in the strip $(0, -i\beta)$ of the complex time plane. Last one formally allows to introduce the Kubo-Martin-Schwinger (KMS) [10] boundary condition:

$$\Phi(T_-) = \Phi(T_- - i\beta). \quad (2.26)$$

This is the last ingredient of the Schwinger's real-time formalism [11]. Note that the temperature introduced through the boundary condition (2.26).

We can consider the limit $T_\pm \rightarrow \pm\infty$. In the perturbation framework the contributions from contours C_2 and C_4 disappear in this limit and choosing $\alpha = -(\varepsilon + \beta)$ we will find the representation (2.12) from (2.22).

This demonstrates the equivalence of the S -matrix theory and of the traditional approach to thermodynamics if (i) the perturbation theory is used and if (ii) the infinite time interval ($T_+ = +\infty$) is considered. Last one is important since only in this limit we are able to see that addition of C_2 contour does not give a new contribution.

In the S -matrix formalism we had *two* boundary conditions (2.10). The second boundary condition of (2.10) should be changed on the contour C_2 in the closed-time path formalism. In result fields are defined on the contour $C_\beta(T_-, T_+)$.

Therefore, under above mentioned constraints we find the equality:

$$R_S(\beta) = R_{NS}(\beta)|_{T_\pm=\pm\infty}, \quad (2.27)$$

So, $R_S(\beta)$ can be written in the “closed-time path” form (2.22), i.e. can be defined on the Matsubara imaginary-time contour.

3 Concluding remarks

Now it is useful to check up the constraints under which the equality (2.27) is valid.

A. Factorization.

The eq.(2.1) fixes the statement that one can find the operator which maps the “mechanical” system of interacting fields on the thermodynamical state. The eq.(2.1) was derived in [1] adopting the reduction formulae to the path-integral formalism. Noting presence of the reduction formulae criticism it must be mentioned that this formulae works well at least in the perturbation theory.

It must be mentioned also that we can introduce the S -matrix as the assumption and then to check this assumption at the very end of calculations. One can find, for instance, that the (hidden) conservation laws lead to the trivial generating functional $R_S = R_0(0)$. This means that the auxiliary (external) field ϕ can not interact with field Φ through the potential $V(\Phi + \phi)$, see (2.7) (the field Φ is “confined” in this case).

The S -matrix approach gives a possibility to investigate this solution. It is interesting to find this possibility in the “closed-time path” formalism also (the problem is connected with boundary condition (2.26) since it hiddely assumes that the field Φ is not confined: $1/\beta$ is the temperature of the interacting fields Φ system).

B. Absence of correlations in the environment.

We had leave only first term in the expansion (2.3) of operator N . If there is the correlations among particles of the environment the higher powers of $(\hat{\phi}_i^* \hat{\phi}_j)$ must be taken into account. In this case the theory will contain arbitrary number of phenomenological parameters. The analogouse realisation of Schwinger’s formalism was offered in [12]. But there is the difference between S -matrix and this generalized approach.

In the S -matrix formalism the operator N contains the only product $(\hat{\phi}_i^* \hat{\phi}_j)$. This allows to interpret $(\hat{\phi}_i^* \hat{\phi}_j)$ as the operator of particles number. Indeed, $\hat{\phi}_i^*$ is the creation operator and $\hat{\phi}_i$ is the absorption one. Therefore, the eigenvalue of $(\hat{\phi}_i^* \hat{\phi}_j)$ is the number of particles [1].

This physical interpretation allows the to introduce the correlations in the “bootstrap” manner considering the system under investigation as the part of a “big” system. In this case all parameters of the S -matrix theory generating functional (2.1) will be fixed.

C. Perturbation theory.

The eq.(2.11) was derived in the perturbation theory framework. The generalization of this formulae on the case of nonperturbate contributions is the easy task considering fields on the real-time contours C_{\pm} . One can use for this purpose the stationary phase method if $Z(\phi_{\pm})$ are calculated, or the unitary formalism [13] if R_0 is calculated (last quantity is preferable if the closed-path boundary conditions [1] shall be used).

But if the fields are defined on the time contour $C_{\beta}(T_-, T_+)$, with both the real- and the imaginary-time parts, the introduction of nonperturbative contributions is the hard problem since it needs the definition of Green functions $G(t, t')$ with, for instance, $t \in C_1$ and $t' \in C_2$.

D. *The equilibrium condition.*

Since $F(\beta_+, \beta_-)$ is the essentially nonlinear function:

$$F(\beta_+, \beta_-) = \sum_{n_+, n_- = 0}^{\infty} \frac{(\beta_+ - \beta)^{n_+}}{n_+!} \frac{(\beta_- - \beta)^{n_-}}{n_-!} F^{(n_+, n_-)}(\beta), \quad (3.1)$$

the definition of integrals (2.22) on the Gauss measure leads to the asymptotic series. The coefficient of the expansion grows

$$\sim \frac{\Gamma((1 + \sum_{k=1}^{\infty} k n_k)/2)}{n_3! n_4! \dots}, \quad (3.2)$$

where the summation over all $n_k \geq 0$, $k = 3, 4, \dots$ is assumed.

The existence of this asymptotic series in the Borel sense depends from the location of singularities over β of the $R_S(\beta)$. We had assume that this series exist by the following reason.

There is the general statement of statistical physics that the canonical (Gibbs) and the microcanonical descriptions are equivalent (at least for the equilibrium case). Following to this statement there are two possibilities.

(i). If the descussed asymptotic series can not be defined then we must postulate that

$$F^{(n_+, n_-)}(\beta) \equiv \frac{\partial^{n_+ + n_-}}{\partial^{n_+} \beta_+ \partial^{n_-} \beta_-} F(\beta_+, \beta_-)|_{\beta_{\pm} = \beta} \equiv 0, n_+ + n_- \geq 3 \quad (3.3)$$

since in opposite case the canonical and the microcanonical descriptions should not coincide. But it is the too strong constraint for interacting fields.

(ii). If the asymptotic series can be “regularised” in the Borel sense than it is sufficient to assume the asymptotic condition:

$$F^{(n_+, n_-)}(\beta) \sim 0, n_+ + n_- \geq 3 \quad (3.4)$$

It is weaker constraint which also guaranties the Gaussian fluctuations of β_{\pm} . Inserting (2.16) into (3.4) one can easily find that (3.4) means smallness of $(n_+ + n_-) \geq 3$ -point energy correlation functions in comparision with 2-point energy correlation function. Note that the condition (3.4) leads to the well known Boltzman’s two-particle approximation. This formal remark can be used as the support of above mentioned assumption that the asymptotic series exist in the Borel sense.

Note that the constraints (3.4) can be measured experimentally. It is important in the particles physics for investigation of the created quark-gluon plasma state.

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References

- [1] J.Manjavidze,*Preprint*,**IP GAS-HE-5/95, (hep-ph/9506424)**(1995)
- [2] T.Matsubara,*Prog.Theor.Phys.***14**,351(1955)
- [3] J.Manjavidze,*Preprint*,**IP GAS-HE-6/95 (hep-ph/9510251)** (1995)
- [4] E.P.Wigner,*Phys.Rev.*,**40**, 749 (1932)
- [5] P.Carrusers and F.Zachariasen, *Pfys.Rev.*, **D13**, 950 (1986)
- [6] R.Mills,*Propagators for Many-Particles Systems* (Gordon and Breach, Science, 1970)
- [7] P.M.Bakshi and K.T.Mohanthappa, *Journ.Math.Phys.*, **4**,1 (1961), *ibid*, **4**, 12 (1961)
- [8] A.J.Niemi and G.Semenoff,*Ann.Phys.(N.Y.)*, **152**, 105 (1984)
- [9] H.Matsumoto, Y.Nakano and H.Umetzava, *J.Math.Phys.*,**25**, 3076 (1984)
- [10] R.Kubo,*J.Phys.Soc.Jap.*, **12**, 570 (1957), M.Martin and J.Schwinger, *Phys.Rev.*, **115**, 342 (1959)
- [11] J.Schwinger, *Particles, Sources and Fields Vol.1* (Addison-Wesley Publ.Comp., 1970)
- [12] E.Calsetta and B.L.Hu, *Phys.Rev.*, **D37**, 2878 (1988)
- [13] J.Manjavidze, *Preprint*, **IP GAS-HE-7/95 (quant-ph/9507003)**(1995)